Al-Ahgaff University Journal of Computer Science and Mathematics
مجلة جامعة الأحقاف لعلوم الحاسوب والرياضيات

Vol. 2, October 2024, pp. 43~51

An Improved Dejdumrong Polynomial Solutions of Systems of Second Order Delay Differential Equations with Proportional and Constant Arguments

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Article Info

Article history:

Received August 01, 2024
Accepted September 02, 2024

Keywords:

Dejdumrong Polynomial
Differential Equations
Initial Value Problem.

ABSTRACT

In this article, a matrix-collocation method is applied to find out the approximate solutions of second-order delay differential equation systems by using the Dejdumrong polynomial. In the method, the functions and the coefficients in the problem are represented in matrix form, and the problem is reduced into an algebraic equation system. Solving this system allows for the determination of the approximation solution's unknown coefficients. Additionally, two examples are used, together having residual error analysis, to demonstrate the effectiveness of the approach. The numerical results show that the obtained solutions exhibit a high level of concurrence with earlier studies.

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في هذه المقالة، يتم تطبيق طريقة تجميع المصفوفات لإيجاد الحلول التقريبية لأنظمة المعادلات التفاضلية المتأخرة من الدرجة الثانية باستخدام متعددة حدود ديجومرونج. في هذه الطريقة، يتم تمثيل الدوال والمعاملات في شكل مصفوفة، ويتم تحويلها إلى نظام معادلات جبرية. يسمح حل هذا النظام بتحديد معاملات الحل التقريبي غير المعروفة. بالإضافة إلى ذلك، يتم استخدام مثالين، مع تحليل الخطأ المتبقي معًا، لإثبات فعالية الطريقة. تُظهر النتائج العددية أن الحلول التي تم الحصول عليها تظهر مستوى عالٍ من التوافق مع الدراسات السابقة.

11. INTRODUCTION

In the current research, we provide an innovative method using Dejdumrong polynomials as a collocation technique for obtaining the solution of a system of linear second-order delay differential equations (SDDEs) expressed in the given form.

$$\sum_{k=0}^{m}\sum_{s=1}^{S} P_{rs}^{k}(\tau)u_{s}^{(k)}(a_{sk}\tau+b_{sk})=h_{r}(\tau), r=1,2,\ldots,S, 0\le a\le\tau\le b \tag{1}$$

under the mixed conditions

$$u_{s}^{(k)}(a)=\mu_{sk}, s=1,2,\ldots,S, k=0,1,\ldots,m-1. \tag{2}$$

The delay differential equations systems play a significant role in the modeling of population dynamics, including predator-prey relationships and HIV infection of CD4+ T cells [1-4]. In recent times, these systems have also kept up with sustainable development for particular models, including medication therapy for HIV infection using the system of fractional differential equations [5]. It is acceptable to express that the analytical solutions of SDDEs are barely obtained via any analytical procedure. Thus, some efficient numerical methods are prepared to consider this issue. So far, a Chebyshev polynomial approach has been suggested as a solution for systems of higher-order differential equations [6]. The Taylor collocation method has been used for linear differential difference equations [7]. The Said-Ball collocation method has been used to solve SDDEs [8]. His method of variational iteration has been used to solve systems of differential equations

[9]. The use of the exponential Galerkin technique and the optimum perturbation iteration method has been suggested as potential approaches for addressing the HIV infection model pertaining to CD4+ T-cells [10, 11]. The optimal perturbation iteration method has also been applied to approach the solution of a fractional Ebola virus disease model [12]. The Adomian decomposition method has been utilized to solve the continuous population models for single and interacting species [13]. The use of the differential transformation approach has been employed in order to derive solutions for systems of differential equations [14].

Having been motivated by the studies above, this study deals with an inventive numerical method based on the Dejdumrong polynomials to readily solve SDDEs, collaborating the matrix expansions concerning the terms in Eq. (1).

This study is organized as follows: Section 2 reveals some properties of the Dejdumrong polynomials. Section 3 comprises the solution method via the collaboration of the matrices. Section 4 is devoted to the error analysis . The accuracy and efficacy of the proposed method are shown by processing two numerical examples in Section 5. Section 6 discusses the innovations and outcomes produced by the method presented in Section 5.

12. REPRESENTATION OF DEJDUMRONG POLYNOMIAL:

A degree m polynomial can be expressed explicitly as [15, 16]

$$\mathcal{D}_i^m(\tau) = \begin{cases} (3\tau)^i(1-\tau)^{i+3}, & \text{for} \quad 0 \le i < \left\lceil \frac{m}{2} \right\rceil - 1, \\ (3\tau)^i(1-\tau)^{m-i}, & \text{if} \quad i = \left\lceil \frac{m}{2} \right\rceil - 1, \\ 2 \cdot 3^{i-1}(1-\tau)^i\tau^i, & \text{if} \quad i \quad \text{is even and} \quad i = \frac{m}{2}, \\ \mathcal{D}_{m-i}^m(1-\tau), & \text{for} \quad \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m. \end{cases} \tag{3}$$

Definition:

The monomial matrix of Dejdumrong polynomial is [15, 16]

$$\mathcal{N} = \begin{bmatrix} n_{00} & n_{01} & \cdots & \cdots & n_{0m} \\ n_{10} & n_{11} & \cdots & \cdots & n_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ n_{m0} & n_{m1} & \cdots & \cdots & n_{mm} \end{bmatrix}_{(m+1)\times(m+1)} \tag{4}$$

where n_{kl} is defined as

$$n_{rs} = \begin{cases} (-1)^{(s-r)}3^r\binom{r+3}{s-r}, & \text{for } 0 \le r \le \left\lceil \frac{m}{2} \right\rceil - 1, \\ (-1)^{(s-r)}3^r\binom{m-r}{s-r}, & \text{for r} = \left\lceil \frac{m}{2} \right\rceil - 1, \\ (-1)^{(s-r)}2(3^{r-1})\binom{r}{s-r}, & \text{for r} = \frac{m}{2} \text{ and } m \text{ is even}, \\ (-1)^{(s-r)}3^{m-r}\binom{m-r}{s-r}, & \text{for r} = \left\lfloor \frac{m}{2} \right\rfloor + 1, \\ (-1)^{(s-m+r-1)}3^{m-r}\binom{m-r}{s-m+r-3}, & \text{for } \left\lfloor \frac{m}{2} \right\rfloor + 1 \le r \le m. \end{cases} \tag{5}$$

$\lfloor \tau \rfloor$ represents the greatest integer $\le \tau$, and $\lceil \tau \rceil$ indicates the least integer $\ge \tau$. The following properties are holding for the Dejdumrong basis function

i. Each basis function of the Dejdumrong is non-negative *i.e* ,

$$\mathcal{D}_i^m(\tau) \ge 0, \forall i = 0,1,\cdots,m.$$

ii. The partition of unity, *i.e*

$$\sum_{i=0}^{m} \mathcal{D}_i^m(\tau) = 1.$$

13. SOLUTION METHOD:

We can write any function $u(\tau)$ by using the first $(m+1)$ terms of Dejdumrong polynomials as follows:

$$u_j(\tau) = \sum_{i=0}^{N} \mathcal{D}_i^N(\tau)c_{ji} = \mathcal{D}^N(\tau)\, C_j = T(\tau)\mathcal{N}C_j, \qquad j = 1,2,\cdots,J. \tag{6}$$

where $\mathcal{N}$ is given in Eq.(4), $T(t) = [1 \quad \tau \quad \tau^2 \quad \dots \quad \tau^N]$ and $C_j = [c_{j0} \quad c_{j1} \quad \cdots \quad c_{jN}]^T$.

However, it is clear that the relation between the k^{th} derivative $T^{(k)}(\tau)$ of $T(\tau)$ is obtained by

$$T^{(k)}(\tau) = T(\tau)\Lambda^k, \qquad k = 1,2,\cdots \tag{7}$$

where

$$\Lambda = \begin{cases} i, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Therefore, using Eq. (7) in Eq. (6) yields

$$u_j^{(k)}(\tau) = T(\tau)\Lambda^k \mathcal{N} C_j, \qquad j = 1,2,\cdots,J. \tag{9}$$

In same manner, if we replace $\tau \to a_{jk}\tau + b_{jk}$ into $Eq.$(9) we get the following relation

$$u_j^{(k)}(a_{jk}\tau + b_{jk}) = T(a_{jk}\tau + b_{jk})\Lambda^k \mathcal{N} C_j = T(\tau)\mathcal{B}(a_{jk}, b_{jk})\Lambda^k \mathcal{N} C_j \tag{10}$$

If $a_{jk} \neq 0$ and $b_{jk} \neq 0$

$$\mathcal{B}(a_{jk}, b_{jk}) = \begin{bmatrix} \binom{0}{0}(a_{jk})^0(b_{jk})^0 & \binom{1}{0}(a_{jk})^0(b_{jk})^1 & \cdots & \cdots & \binom{N}{0}(a_{jk})^0(b_{jk})^N \\ 0 & \binom{1}{1}(a_{jk})^1(b_{jk})^0 & \cdots & \cdots & \binom{N}{1}(a_{jk})^1(b_{jk})^{N-1} \\ 0 & 0 & \ddots & \cdots & \binom{N}{2}(a_{jk})^2(b_{jk})^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(a_{jk})^N(b_{jk})^0 \end{bmatrix}$$

Now by substituting Eq.(6), Eq. (9) and Eq. (10) in Eq. (1), Then we can express Eq. (1) in matrix form as

$$\sum_{k=0}^{m} \boldsymbol{P}_k \boldsymbol{u}^{(k)}(a_k\tau + b_k) = H(\tau) \tag{11}$$

where

$$\boldsymbol{u}^{(k)}(a_k\tau + b_k) = \begin{bmatrix} u_1^{(k)}(a_{1k}\tau + b_{1k}) \\ u_2^{(k)}(a_{2k}\tau + b_{2k}) \\ \vdots \\ u_{jk}^{(k)}(a_{jk}\tau + b_{jk}) \end{bmatrix} = \begin{bmatrix} T(\tau)\mathcal{B}(a_{1k}, b_{1k})\Lambda^k \mathcal{N} C_1 \\ T(\tau)\mathcal{B}(a_{2k}, b_{2k})\Lambda^k \mathcal{N} C_2 \\ \vdots \\ T(\tau)\mathcal{B}(a_{jk}, b_{jk})\Lambda^k \mathcal{N} C_j \end{bmatrix} = \overline{T}(\tau)\overline{\mathcal{B}}(a_k, b_k)\left(\overline{\Lambda}\right)^k \overline{\mathcal{N}}\boldsymbol{C},$$

$$\overline{\mathcal{B}}(a_k, b_k) = \begin{bmatrix} \mathcal{B}(a_k, b_k) & 0 & \cdots & 0 \\ 0 & \mathcal{B}(a_k, b_k) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{B}(a_k, b_k) \end{bmatrix}, \quad \overline{\Lambda}^k = \begin{bmatrix} \Lambda^k & 0 & \cdots & 0 \\ 0 & \Lambda^k & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \Lambda^k \end{bmatrix},$$

$$\overline{T}(\tau) = \begin{bmatrix} T(\tau) & 0 & \cdots & 0 \\ 0 & T(\tau) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & T(\tau) \end{bmatrix}, \quad \overline{\mathcal{N}} = \begin{bmatrix} \mathcal{N} & 0 & \cdots & 0 \\ 0 & \mathcal{N} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{N} \end{bmatrix},$$

$$\boldsymbol{P}_k(\tau) = \begin{bmatrix} p_{11}^k(\tau) & p_{12}^k(\tau) & \cdots & p_{1J}^k(\tau) \\ p_{21}^k(\tau) & p_{22}^k(\tau) & \cdots & p_{2J}^k(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ p_{J1}^k(\tau) & p_{J2}^k(\tau) & \cdots & p_{JJ}^k(\tau) \end{bmatrix}, \qquad \boldsymbol{H}(\tau) = \begin{bmatrix} h_2(\tau) \\ h_2(\tau) \\ \vdots \\ h_J(\tau) \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_J \end{bmatrix}.$$

By using the collocation points given by

$$\tau_s = a + \frac{b-a}{N}s, s = 0,1,\ldots,N. \tag{12}$$

Substituting Eq. (12) in Eq. (11), it is obtained the system of the matrix equations

$$\sum_{k=0}^{m} \overline{\boldsymbol{P}_k}(\tau)\,\overline{\boldsymbol{T}}(\tau_s)\overline{\mathcal{B}}(a_k, b_k)\left(\overline{\Lambda}\right)^k \overline{\mathcal{N}} C = \boldsymbol{H}(t_s) \tag{13}$$

where

$$\overline{\boldsymbol{P}_k}=\begin{bmatrix}\boldsymbol{P}_k(\tau_0) & 0 & \cdots & 0\\ 0 & \boldsymbol{P}_k(\tau_1) & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \boldsymbol{P}_k(\tau_N)\end{bmatrix},\ \overline{\boldsymbol{H}}=\begin{bmatrix}\boldsymbol{H}(\tau_0)\\ \boldsymbol{H}(\tau_1)\\ \vdots\\ \boldsymbol{H}(\tau_N)\end{bmatrix}\text{ and}$$

$$\overline{\boldsymbol{T}}=\begin{bmatrix}\boldsymbol{T}(\tau_0) & 0 & \cdots & 0\\ 0 & \boldsymbol{T}(\tau_1) & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \boldsymbol{T}(\tau_N)\end{bmatrix}.$$

The fundamental matrix Eq. (13) for Eq. (1) generates a system of $k(N+1)$ algebraic equations that can be solved to obtain the $k(N+1)$ unknown Dejdumrong coefficients

$$\boldsymbol{WC}=\boldsymbol{H}\text{ or }[\boldsymbol{W};\boldsymbol{H}] \tag{14}$$

where

$$\boldsymbol{W}=\sum_{k=0}^{m}\boldsymbol{P}_k(\tau_s)\,\overline{\boldsymbol{T}}(\tau_s)\overline{\mathcal{B}}(a_k,b_k)\left(\overline{\Lambda}\right)^k\overline{\mathcal{N}}$$

By using the Eq. (9), yields the conditions Eq. (2) in matrix form for $j=1,2,\dots,J,\ k=0,1,\dots,m-1$ as follows:

$$\begin{bmatrix}u_1^{(k)}(a)\\ u_2^{(k)}(a)\\ \vdots\\ u_J^{(k)}(a)\end{bmatrix}=\begin{bmatrix}T(a)\Lambda^k\mathcal{N} & 0 & \cdots & 0\\ 0 & T(a)\Lambda^k\mathcal{N} & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & T(a)\Lambda^k\mathcal{N}\end{bmatrix}\begin{bmatrix}C_1\\ C_2\\ \vdots\\ C_J\end{bmatrix}=\begin{bmatrix}\mu_{1k}\\ \mu_{2k}\\ \vdots\\ \mu_{Jk}\end{bmatrix}$$

Or simply

$$\boldsymbol{U}_k\boldsymbol{C}=\mu_k\text{ or }[\boldsymbol{U}_k;\mu_k],\qquad k=0,1,\cdots,m-1. \tag{15}$$

Therefore, the conditional row of the matrices (15) is swapped out with the last rows of the matrix (14), and we receive the new augmented matrix, which appears as

$$\widetilde{\boldsymbol{W}}\boldsymbol{C}=\widetilde{\boldsymbol{H}}\text{ or }\left[\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{H}}\right]. \tag{16}$$

If $rank(\widetilde{\boldsymbol{W}})=rank\left(\left[\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{H}}\right]\right)=k(N+1)$, then, we can write

$$\boldsymbol{C}=\left(\widetilde{\boldsymbol{W}}\right)^{-1}\widetilde{\boldsymbol{H}}.$$

So, the matrix C is determined uniquely, and the Eq. (1) under the coefficient equation Eq. (2) has a unique solution. This solution is provided by Dejdumrong polynomial

$$u_j(\tau)\cong u_{j,N}(\tau)=\sum_{n=0}^{N}c_{jn}\mathcal{D}_i^N(\tau).$$

4. RESIDUAL ERROR ESTIMATION AND IMPROVEMENT OF SOLUTIONS

The actual error functions are defined as

$$e_{j,N}(\tau)=u_j(\tau)-u_{j,N}(\tau),\qquad j=1,2,\dots,k$$

where $u_{j,N}(\tau)$ and $u_j(\tau)$, represent the approximate solutions and the exact solutions respectively.

Let's first define the residual function. The Dejdumrong polynomial solutions are expressed in the system (1) as

$$R_{i,N}(\tau)=\sum_{j=1}^{k}P_{2,j}(\tau)u_{j,N}^{(2)}(\tau)+\sum_{j=1}^{k}P_{1,j}(\tau)u_{j,N}^{(1)}(\tau)+\sum_{j=1}^{k}P_{0j}(\tau)u_{j,N}(\tau)=h_i(\tau),i=1,2,\dots,k.$$

or briefly, we can write as

$$\sum_{j=1}^{k}P_{2,j}(\tau)u_{j,N}^{(2)}(\tau)+\sum_{j=1}^{k}P_{1,j}(\tau)u_{j,N}^{(1)}(\tau)+\sum_{j=1}^{k}P_{0j}(\tau)u_{j,N}(\tau)=h_i(\tau)-R_{i,N}(\tau). \tag{17}$$

On same manner, The Dejdumrong polynomial solutions are expressed in condition Eq. (2) as

$$\sum_{i=1}^{k}u_{i,N}^{(k)}(a)=\mu_i,i=1,2,\dots,k. \tag{18}$$

Secondly, the system of error differential equations is obtained as if Eq. (17) is subtracted from Eq. (1).

$$\sum_{j=1}^{k}P_{2,j}(\tau)\left[u_j^{(2)}(\tau)-u_{j,N}^{(2)}(\tau)\right]+\sum_{j=1}^{k}P_{1,j}(\tau)\left[u_j^{(1)}(\tau)-u_{j,N}^{(1)}(\tau)\right]+\sum_{j=1}^{k}P_{0j}(\tau)\left[u_j(\tau)-u_{j,N}(\tau)\right]=-R_{i,N}(\tau).$$

instead, we may use the short form

$$\sum_{j=1}^{k} P_{2,j}(\tau)\left[e_{j,N}^{(2)}(\tau)\right] + \sum_{j=1}^{k} P_{1,j}(\tau)\left[e_{j,N}^{(1)}(\tau)\right] + \sum_{j=1}^{k} P_{0j}(\tau)\left[e_{j,N}(\tau)\right] = -R_{i,N}(\tau).$$

Afterward likewise, if Eq. (18) is subtracted from Eq. (2), then we obtain in the form

$$\sum_{i=1}^{k}\left(u_i^{(r)}(a) - u_{i,N}^{(r)}(a)\right) = \mu_i, r = 0, 1, \dots, m-1.$$

we might also use the short form

$$\sum_{i=1}^{k} e_{i,N}^{(r)}(a) = \mu_i, r = 0,1, \dots, m-1. \tag{19}$$

As a result, we arrive at approximations of solutions to the error problems Eq. (18) and Eq. (19)in the form

$$e_{i,N,M} = \sum_{n=0}^{M} \widehat{a}_{i,n}\, \mathcal{D}_n(\tau), i = 1,2, \dots, k.$$

Therefore, by the amalgamation of the approximate solution to the error issue and the approximate solution itself, we are able to derive an improved approximation in the specified format.

$$u_{i,N,M}(\tau) = u_{i,N}(\tau) + e_{i,N,M}.$$

Finally, we get the error function of the improved approximate solution in the form.

$$E_{i,N,M}(\tau) = u_i(\tau) - u_{i,N,M}(\tau).$$

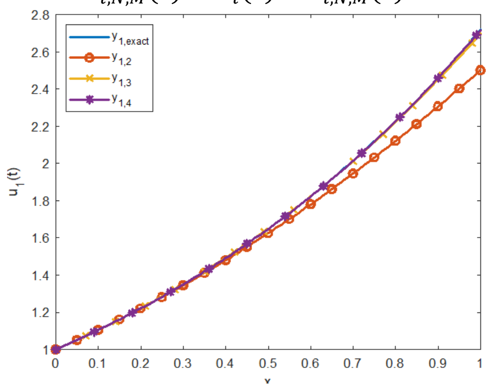

Figure 11. Comparing approximation solution numerical results with various $\boldsymbol{N}$ values for $\boldsymbol{u_1(\tau)}$.

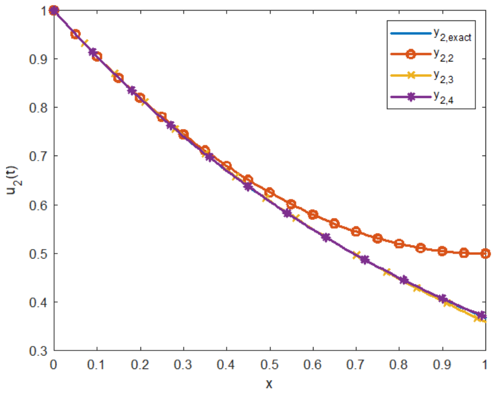

Figure 12. Comparing approximation solution numerical results with various $\boldsymbol{N}$ values for $\boldsymbol{u_2(\tau)}$.

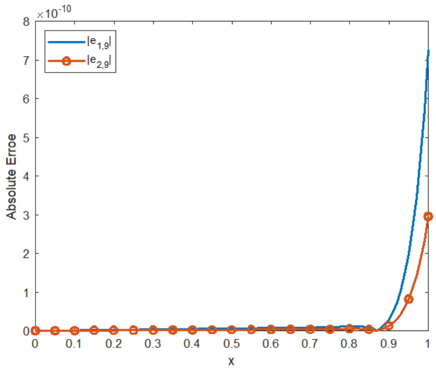

Figure 13. Comparison of the absolute errors of $\boldsymbol{u_1(\tau)}$ and $\boldsymbol{u_2(\tau)}$ Example 2

5. RESULTS AND DISCUSSION

This section uses two illustrated examples to demonstrate the method's effectiveness and accuracy. In that case, a computer program on Matlab R2021a is developed following the mainframe of the method. The Dejdumrong polynomial solutions $u_{j,N}$, corrected Dejdumrong polynomial solutions $u_{j,N,M} = u_{j,N}(\tau) + e_{j,N,M}(\tau)$ and error function $e_{j,N}(\tau) = u_j(\tau) - u_{j,N}(\tau)$ are calculated. As a result, tables and figures directly summarize and compare the computation results of the examples.

Example 1: First, we are considering the system of linear delay-differential equations [17]

$$\begin{cases} u_1^{(2)}(\tau) + tu_1(\tau-1) + \tau u_2(\tau) = 2\tau^2 - \tau^3 - 2, \\ u_2^{(2)}(\tau) + 2\tau u_2(\tau-1) + 2\tau u_1(\tau) = 4\tau^2 - 2\tau^3. \end{cases} \tag{20}$$

and the initial conditions $u_1(0) = 0, u_2(0) = 1, u_1^{(1)}(0) = 1$ and $u_2^{(1)}(0) = 1$ with the exact solutions are $u_1(\tau) = \tau - \tau^2, u_2(\tau) = \tau + 1$. for $N = 2$, the approximate solutions $u_j(\tau), j = 1,2$ by the Dejdumrong polynomial

$$u_{j,2}(\tau) = \sum_{n=0}^{2} c_{jn}\mathcal{D}_i^2(\tau).$$

Firstly, the set of collocation points are $\left\{\tau_0 = 0, \tau_1 = \frac{1}{2}, \tau_2 = 1\right\}$, and from Eq. (14), we write the fundamental matrix equation of the problem as

$$\left\{P\overline{T}\overline{\Lambda}^2 + \beta_1\overline{T}\mathcal{B}(1,-1) + \beta_2\overline{T}\mathcal{B}(1,0)\right\}C = H$$

where

$$P = \begin{bmatrix} P(0) & 0 & 0 \\ 0 & P(\frac{1}{2}) & 0 \\ 0 & 0 & P(1) \end{bmatrix}, P(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \beta_1 = \begin{bmatrix} \beta_1(0) & 0 & 0 \\ 0 & \beta_1(\frac{1}{2}) & 0 \\ 0 & 0 & \beta_1(1) \end{bmatrix}, \beta_1(\tau) = \begin{bmatrix} \tau & 0 \\ 0 & 2\tau \end{bmatrix}, \beta_2$$

$$= \begin{bmatrix} \beta_2(0) & 0 & 0 \\ 0 & \beta_2(\frac{1}{2}) & 0 \\ 0 & 0 & \beta_2(1) \end{bmatrix}, \beta_1(\tau) = \begin{bmatrix} 0 & \tau \\ 2\tau & 0 \end{bmatrix}$$

$$\overline{T}(\tau) = \begin{bmatrix} T(\tau) & 0 \\ 0 & T(\tau) \end{bmatrix}, \overline{T} = \begin{bmatrix} \overline{T}(0) & 0 & 0 \\ 0 & \overline{T}(\frac{1}{2}) & 0 \\ 0 & 0 & \overline{T}(1) \end{bmatrix}$$

$$\overline{\Lambda} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\mathcal{B}}(1,-1) = \begin{bmatrix} \mathcal{B}(1,-1) & 0 \\ 0 & \mathcal{B}(1,-1) \end{bmatrix}, \mathcal{B}(1,-1) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overline{\mathcal{B}}(1,0) = \begin{bmatrix} \mathcal{B}(1,0) & 0 \\ 0 & \mathcal{B}(1,0) \end{bmatrix}, \mathcal{B}(1,-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} 2 & -4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 2 \\ 25/8 & -19/4 & 17/8 & 1/8 & 1/4 & 1/8 \\ 1/4 & 1/2 & 1/4 & 17/4 & -11/2 & 9/4 \\ 3 & -4 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 4 & -4 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{bmatrix}$$

Replace the last four rows of W by U yields

$$[\widetilde{W};\tilde{G}] = \begin{bmatrix} 2 & -4 & 2 & 0 & 0 & 0 & ; & -2 \\ 0 & 0 & 0 & 2 & -4 & 2 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 1 \\ -2 & 2 & 0 & 0 & 0 & 0 & ; & 1 \\ 0 & 0 & 0 & -2 & 2 & 0 & ; & 1 \end{bmatrix}, \text{ Therefore } C = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \\ 3/2 \\ 2 \end{bmatrix}.$$

Hence,

$u_1(\tau) = \tau - \tau^2$ and $u_2(\tau) = \tau + 1$. are the approximate solution which is similar to the exact solution.

Example 4.2. Consider a system of second order delay differential equations [18]

$$\begin{cases} u_1^{(2)}(\tau) - u_1(\tau) + u_2(\tau) - u_1(\tau - 0.2) = -e^{\tau-0.2} + e^{-\tau}, \\ u_2^{(2)}(\tau) + u_1(\tau) - u_2(\tau) - u_2(\tau - 0.2) = -e^{-\tau+0.2} + e^{\tau}, \end{cases} \tag{21}$$

Subject to the initial conditions

$$u_1(0) = u_1^{(1)}(0) = 1, u_2(0) = 1 \text{ and } u_2^{(1)}(0) = -1.$$

The exact solution of the problem is $u_1(\tau) = e^{\tau}$ and $u_2(\tau) = e^{-\tau}$.

Tables 1 and 2 compare the absolute errors to [10] with the suggested method at $N = 10$ and $M = 12$. Tables 3 and 4 display absolute errors and correction errors discovered using the proposed technique and [10, 19] for $N = 14$ and $M = 16$. However, Figures 1 and 2 depict the approximations for various values of N. It is evident that as N increases, the approximate solutions derived by the method used closely align with the exact solutions, while Figure 3 displays the absolute error for $u_1(\tau)$ and $u_2(\tau)$ when $N = 9$.

Table 5. Comparison of the absolute errors $\boldsymbol{e_N}$ for $\boldsymbol{N = 10}$ for $\boldsymbol{u_1(\tau)}$ of $\boldsymbol{Eq. (21)}$

τ_i	Ref [19]	Ref [18]	PM	
		$\theta = \nu = -1/2$	$e_{1,10}$	$e_{1,10,12}$
0.1	4.3E-6	7.5884E-14	1.02637e-15	1.28826e-16
0.2	2.8E-5	2.1963E-13	1.72420e-15	3.16876e-16
0.3	8.1E-5	3.1027E-13	1.06733e-15	5.10688e-16
0.4	3.0E-4	4.5289 E-13	2.80113e-15	7.14267e-16
0.5	7.3E-4	6.3948 E-13	4.59342e-15	9.31397e-16

Table 6. Comparison of the absolute errors $\boldsymbol{e_N}$ for $\boldsymbol{N = 10}$ for $\boldsymbol{u_2(\tau)}$ of $\boldsymbol{Eq. (21)}$

τ_i	Ref [10]	Ref [18]	PM	
		$\theta = \nu = -1/2$	$e_{2,10}$	$e_{2,10,12}$
0.1	4.1E-6	3.5036 E-14	1.59799E-15	6.23645e-17
0.2	2.2E-5	1.0140 E-13	1.77660E-15	1.54125e-16
0.3	4.5E-5	1.4675 E-13	4.61457E-15	2.51117e-16
0.4	1.4E-6	2.1655 E-13	6.98641E-15	3.56744e-16
0.5	2.6E-4	3.0890 E-13	9.48258E-15	4.74490e-16

Table 7. Comparison of the absolute errors $\boldsymbol{e_N}$ *for* $\boldsymbol{N=14}$ *for* $\boldsymbol{u_1(\tau)}$ *of* $\boldsymbol{Eq.(21)}$

τ_i	Ref [18]		PM	
	$\theta=\nu=0$	$\theta=\nu=1/2$	$e_{1,14}$	$e_{1,14,16}$
0.1	1.2882E-16	9.2420E-17	3.70057E-19	6.84999e-22
0.2	2.6693E-18	6.3736E-18	8.74132E-19	1.63757e-21
0.3	7.5158E-17	1.1398E-16	8.19378E-19	2.62505e-21
0.4	1.1714E-16	3.6547E-17	5.64073E-19	3.66455e-21
0.5	5.2956E-16	1.2615E-16	1.74023E-18	4.78091e-21

Table 8. Comparison of the absolute errors $\boldsymbol{e_N}$ *for* $\boldsymbol{N=14}$ *for* $\boldsymbol{u_2(\tau)}$ *of* $\boldsymbol{Eq.(21)}$

τ_i	Ref [18]		PM	
	$\theta=\nu=0$	$\theta=\nu=1/2$	$e_{2,14}$	$e_{2,14,16}$
0.1	2.3027E-16	1.7771E-17	1.52292E-19	3.07513e-22
0.2	1.8795E-16	3.9806E-17	3.90163E-19	7.37858e-22
0.3	1.5197E-16	6.0848E-17	9.40023E-19	1.19395e-21
0.4	2.5009E-17	1.9433E-16	1.68942E-18	1.69145e-21
0.5	4.6413E-17	1.4602E-16	1.65901E-18	2.25002e-21

Tables 5–6 demonstrate that the corrected absolute errors get closer to zero as M increases. As a result, we can conclude that the Dejdumrong polynomial solutions method is particularly efficient for the system of differential equations (21).

Table 9. Comparison of $\left|\boldsymbol{E_{1,N,M}}\right|$ *corrected absolute errors for* $\boldsymbol{N=3}$ *and* $\boldsymbol{M=4,6,9}$ *of the Eq. (21)*

τ_i	Ref [20]			PM		
	$\left\|E_{1,3,4}\right\|$	$\left\|E_{1,3,6}\right\|$	$\left\|E_{1,3,9}\right\|$	$\left\|E_{1,3,4}\right\|$	$\left\|E_{1,3,6}\right\|$	$\left\|E_{1,3,9}\right\|$
0.0	0.00E+00	0.00E+00	0.00E+00	0.000E+00	0.000E+00	0.000E+00
0.1	2.865E-4	1.492E-6	6.747E-11	2.865E-06	1.492E-08	7.309E-13
0.3	3.607E-3	9.983E-6	1.916E-10	3.607E-05	9.982E-08	3.319E-12
0.5	5.375E-3	1.807E-5	4.171E-10	5.375E-05	1.807E-07	5.870E-12
0.8	3.474E-2	4.510E-5	1.754E-11	3.474E-04	4.509E-07	1.181E-11
1.0	2.074E-1	8.450E-4	7.558E-08	2.074E-03	8.450E-06	7.238E-10

Table 10. Comparison of $\left|\boldsymbol{E_{2,N,M}}\right|$ *corrected absolute errors for* $\boldsymbol{N=3}$ *and* $\boldsymbol{M=4,6,9}$ *of the Eq. (21)*

τ_i	Ref [20]			PM		
	$\left\|E_{2,3,4}\right\|$	$\left\|E_{2,3,6}\right\|$	$\left\|E_{2,3,9}\right\|$	$\left\|E_{2,3,4}\right\|$	$\left\|E_{2,3,6}\right\|$	$\left\|E_{2,3,9}\right\|$
0.0	0.00E+00	0.00E+00	0.00E+00	0.000E+00	0.000E+00	0.000E+00
0.1	1.920E-3	8.381E-7	4.711 E-10	1.920E-06	8.378E-09	3.418E-13
0.3	2.376E-3	5.674E-6	2.280E-09	2.376E-05	5.672E-08	1.541E-12
0.5	3.798E-3	1.055E-5	3.171E-09	3.798E-05	1.055E-07	2.614E-12
0.8	1.931E-2	2.582E-5	2.912E-09	1.931E-04	2.581E-07	4.741E-12
1.0	9.957E-2	3.713E-4	1.288E-08	9.957E-04	3.713E-06	2.954E-10

6. CONCLUSIONS

SDDEs have been suitably solved by the present method based on the Dejdumrong polynomials and the collaboration of the matrices of the terms in the system (1). In that case, a computer program of the method has managed to perform its routines on a considered example. Thus, the numerical and graphic investigations have revealed the broad applicability and practicality of the present approach. The current method sheds light on new aspects and advancements regarding the use of combinatorial polynomials in SDDEs. Thereby, one can admit that the present method is inventive and straightforward in employing the systems of integro-differential-delay and nonlinear differential-delay equations. Of course, some modifications can be implemented.

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